

Factorization technique and isochronous condition for coupled quadratic and mixed Liénard-type nonlinear systems

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Abstract

In this paper, we discuss a systematic and self consistent procedure to factorize a rather general class of coupled nonlinear ordinary differential equations (ODEs), namely coupled quadratic and mixed Liénard type equations, which include various physical and mathematical models. The procedure is broadly divided into two parts. In the first part, we consider a general factorized form for the equation under consideration in terms of some unknown functions and identify the determining equations for them. In the second part, we systematically solve the determining equations and identify the compatible factorizing form for this class of equations. In addition, we also discuss the problem of identification of isochronous dynamical systems belonging to the above class of equations. In particular, we deduce an isochronicity condition for the coupled quadratic Liénard equation. We also present specific examples of physical interest.

1. Introduction

The factorization method is well known in quantum mechanics for solving certain kind of ordinary differential equations (ODEs). It is an operational procedure which enables one to answer, in a direct manner, questions about a class of eigenvalue problems. The underlying idea is to consider a pair of first order differential-difference equations with boundary conditions [1, 2]. For example, decomposition of the quantum linear harmonic oscillator problem in terms of creation and annihilation operators is a case point. Recently, it has been shown by Rosu and his co-workers that at least in the case of some polynomial nonlinearities particular solutions may be found rather simply by an elegant method of factorizing them [3, 4]. This method has been explored in the case of scalar ODEs and nonlinear partial differential equations (PDEs) also and several classes of solutions of many problems have been obtained rather straightforwardly [5, 6, 7, 8]. Further, this method has been applied to the case of a system of coupled Liénard type equations with linear velocity terms and specific classes of Liénard type systems were identified for which particular solutions may be found by solving a Bernoulli equation [9].

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Eventhough, this method plays a crucial role in understanding the nature of the various physical and mathematical models, it is very difficult to obtain the factorized form even in the case of scalar nonlinear ODEs. In the case of coupled ODEs the problem of factorizing the given equation becomes much more complex and one needs a systematic procedure to obtain the factorized form. Our aim, in this paper, is to obtain the factorized form for a general class of coupled ODEs of the type (coupled Liénard type equations with quadratic velocities)

$$\ddot{x} + h_1(x, y)\dot{x}^2 + h_2(x, y)\dot{y}^2 + h_3(x, y)\dot{x}\dot{y} + g_1(x, y) = 0, \quad (1a)$$

$$\ddot{y} + h_4(x, y)\dot{x}^2 + h_5(x, y)\dot{y}^2 + h_6(x, y)\dot{x}\dot{y} + g_2(x, y) = 0, \quad (1b)$$

where h_i 's, $i = 1, \dots, 6$ and g_j 's, $j = 1, 2$ are functions of x and y and then extend this procedure to an even more general class of coupled mixed (quadratic and linear) Liénard type equations

$$\ddot{x} + h_1(x, y)\dot{x}^2 + h_2(x, y)\dot{y}^2 + h_3(x, y)\dot{x}\dot{y} + f_1(x, y)\dot{x} + f_2(x, y)\dot{y} + g_1(x, y) = 0, \quad (2a)$$

$$\ddot{y} + h_4(x, y)\dot{x}^2 + h_5(x, y)\dot{y}^2 + h_6(x, y)\dot{x}\dot{y} + f_3(x, y)\dot{x} + f_4(x, y)\dot{y} + g_2(x, y) = 0. \quad (2b)$$

The above equations include several physically and mathematically important equations and have been studied by many authors. In order to obtain the factorized forms for the above equations, in this paper we develop a systematic and self consistent procedure. For this purpose, we broadly divide our analysis into two parts. In the first part, we consider a general factorized form for Eq. (1) in the form of unknown functions to be determined. In fact an analysis of the scalar version of Eq. (1), namely

$$\ddot{x} + h(x)\dot{x}^2 + g(x) = 0 \quad (3)$$

can itself give important clues [10, 11, 12]. Using this knowledge, expanding and comparing the factorized equation with the original equation (1) we will get a set of PDEs for the coefficients h_i 's and g_j 's which in turn gives a set of determining equations for the unknown functions. To fix the factorized form corresponding to Eq. (1) we need to solve the obtained determining equations for the unknowns. Now, in the second part, we discuss the procedure to solve the set of determining equations consistently for the unknown functions. Solving the determining equations we can obtain the form of the unknowns which in turn will fix the factorized form corresponding to Eq. (1). To illustrate the effectiveness of this procedure we consider the coupled Mathews-Lakshmanan (ML) oscillator equations [13, 14, 15, 16, 17] and show how one can proceed systematically to identify the factorized form. It is to be noted that the study of Liénard type equation carried out by Hazra *et al.* [9] deals with identifying specific classes of Liénard type systems with linear velocity terms only for which particular solutions may be found by solving a Bernoulli equation. However, in this work, we focus our attention in developing a self consistent procedure in order to get the factorized form for the quadratic and mixed Liénard type equation.

In addition to this, we will also discuss the isochronous properties associated with Eq. (1). For this purpose, we unearth an isochronicity condition for Eqs. (1) by transforming our system (Eq. (1)) into a set of uncoupled simple harmonic oscillator equations as the latter set is a prototype of an isochronous system. The obtained isochronicity condition can be used to identify the class of equations exhibiting isochronous properties. We also consider a specific example exhibiting isochronous property. Finally, we include the linear velocity term in addition to quadratic term in Eq. (1) to get an overview of a more general class of equation, that is mixed

Liénard type equation, Eq. (2). We show that the inclusion of the linear velocity term needs a small modification in the procedure discussed for the quadratic Liénard type equation.

The plan of the paper is as follows. In Sec. 2, we consider a general factorized form for Eq. (1) involving a set of unknown functions $\phi_k(x, y)$ and $\psi_{1,2}$, where $k = 1, 2, \dots, 8$. We then develop a systematic algorithm for obtaining the factorized form in Sec. 3. Next, in Sec. 4, we systematically determine the forms of the unknown functions, ϕ_k 's, corresponding to the factorized form of Eq. (1). The forms of the functions $\psi_{1,2}$ are determined in Sec. 5. In Sec. 6, we demonstrate the procedure by considering the coupled ML oscillator equation. We discuss the isochronous property associated with Eq. (1), in Sec. 7. In Sec. 8, we consider a specific example corresponding to isochronous case. The case of mixed Liénard type equation is briefly discussed in Sec. 9. Factorization of Eq. (3) is discussed in Appendix A. In Appendix B, we discuss the factorization of scalar case corresponding to Eq. (2). Finally, our conclusions are given in Sec. 10.

2. Factorization of the general case

To start with, considering the scalar equation (3) it can be factorized as discussed in Appendix A. Taking this as a starting point, let us presume that the coupled quadratic Liénard type equation (1) can be factorized in the form

$$[\phi_7(x, y)D - \phi_1(x, y)][\phi_5(x, y)D - \phi_2(x, y)]\psi_1(x, y) = 0, \quad (4a)$$

$$[\phi_8(x, y)D - \phi_3(x, y)][\phi_6(x, y)D - \phi_4(x, y)]\psi_2(x, y) = 0, \quad (4b)$$

where $D = \frac{d}{dt}$, $\psi_{1,2}$'s and ϕ_k 's, $k = 1, 2, \dots, 8$, are unknown functions of x and y to be determined. Now, the above set of equations can be rewritten as a set of first order coupled differential equations as

$$[\phi_7(x, y)D - \phi_1(x, y)]P_1(x, y) = 0, \quad (5a)$$

$$[\phi_8(x, y)D - \phi_3(x, y)]P_2(x, y) = 0, \quad (5b)$$

$$[\phi_5(x, y)D - \phi_2(x, y)]\psi_1(x, y) = P_1(x, y), \quad (5c)$$

$$[\phi_6(x, y)D - \phi_4(x, y)]\psi_2(x, y) = P_2(x, y). \quad (5d)$$

Hence, our problem of finding the general solution of (1) is converted into simultaneously solving the above set of coupled first order differential equations, provided the decomposition (4) for (1) exists and can be explicitly found. A particular solution of Eq. (1) can be obtained by solving the reduced set of equations,

$$[\phi_5(x, y)D - \phi_2(x, y)]\psi_1(x, y) = 0, \quad (6a)$$

$$[\phi_6(x, y)D - \phi_4(x, y)]\psi_2(x, y) = 0, \quad (6b)$$

which may in some cases be relatively simple.

Now, to identify the forms of the functions h_i and g_j , $i = 1, 2, \dots, 6$ and $j = 1, 2$, for which Eq. (1) can be factorized in the form (4), we expand the latter and compare the resulting form with (1) appropriately for equivalence of (1) with (4). Then equating the governing powers of \dot{x} and \dot{y} , we can show that the various coefficients h_1, h_2, \dots, h_6 and g_1 and g_2 are related to the unknown functions $\phi_1, \phi_2, \dots, \phi_8$ through the relations

$$h_1 = \frac{1}{\delta}[\phi_6\phi_7\phi_8\psi_{2y}(\psi_{1x}\phi_{5x} + \psi_{1xx}\phi_5) - \phi_5\phi_7\phi_8\psi_{1y}(\psi_{2x}\phi_{6x} + \psi_{2xx}\phi_6)], \quad (7a)$$

$$h_2 = \frac{1}{\delta}[\phi_6\phi_7\phi_8\psi_{2y}(\psi_{1y}\phi_{5y} + \psi_{1yy}\phi_5) - \phi_5\phi_7\phi_8\psi_{1y}(\psi_{2y}\phi_{6y} + \psi_{2yy}\phi_6)], \quad (7b)$$

$$h_3 = \frac{1}{\delta} [\phi_6 \phi_7 \phi_8 \psi_{2y} (\psi_{1x} \phi_{5y} + 2\phi_5 \psi_{1xy} + \psi_{1y} \phi_{5x}) - \phi_5 \phi_7 \phi_8 \psi_{1y} (\psi_{2x} \phi_{6y} + 2\phi_6 \psi_{2xy} + \psi_{2y} \phi_{6x})], \quad (7c)$$

$$h_4 = \frac{1}{\delta} [-\phi_6 \phi_7 \phi_8 \psi_{2x} (\psi_{1x} \phi_{5x} + \psi_{1xx} \phi_5) + \phi_5 \phi_7 \phi_8 \psi_{1x} (\psi_{2x} \phi_{6x} + \psi_{2xx} \phi_6)], \quad (7d)$$

$$h_5 = \frac{1}{\delta} [-\phi_6 \phi_7 \phi_8 \psi_{2x} (\psi_{1y} \phi_{5y} + \psi_{1yy} \phi_5) + \phi_5 \phi_7 \phi_8 \psi_{1x} (\psi_{2y} \phi_{6y} + \psi_{2yy} \phi_6)], \quad (7e)$$

$$h_6 = \frac{1}{\delta} [-\phi_6 \phi_7 \phi_8 \psi_{2x} (\psi_{1x} \phi_{5y} + 2\phi_5 \psi_{1xy} + \psi_{1y} \phi_{5x}) + \phi_5 \phi_7 \phi_8 \psi_{1x} (\psi_{2x} \phi_{6y} + 2\phi_6 \psi_{2xy} + \psi_{2y} \phi_{6x})] \quad (7f)$$

and the forms of g_j' s, $j = 1, 2$, as

$$g_1 = \frac{1}{\delta} [\phi_1 \phi_2 \phi_6 \phi_8 \psi_{2y} \psi_1 - \phi_3 \phi_4 \phi_5 \phi_7 \psi_{1y} \psi_2], \quad (8a)$$

$$g_2 = \frac{1}{\delta} [-\phi_1 \phi_2 \phi_6 \phi_8 \psi_{2x} \psi_1 + \phi_3 \phi_4 \phi_5 \phi_7 \psi_{1x} \psi_2], \quad (8b)$$

where the quantity

$$\delta = \phi_5 \phi_6 \phi_7 \phi_8 (\psi_{1x} \psi_{2y} - \psi_{2x} \psi_{1y}) \neq 0. \quad (9)$$

Further, as Eq. (1) does not contain the terms \dot{x} and \dot{y} , their coefficients must be set equal to zero. Consequently we obtain the relations

$$\phi_6 \phi_8 \psi_{2y} [(\phi_7 (\phi_2 \psi_{1x} + \psi_1 \phi_{2x}) + \phi_1 \phi_5 \psi_{1x})] = \phi_5 \phi_7 \psi_{1y} [(\phi_8 (\phi_4 \psi_{2x} + \psi_2 \phi_{4x}) + \phi_3 \phi_6 \psi_{2x})], \quad (10a)$$

$$\phi_6 \phi_8 \psi_{2y} [(\phi_7 (\phi_2 \psi_{1y} + \psi_1 \phi_{2y}) + \phi_1 \phi_5 \psi_{1y})] = \phi_5 \phi_7 \psi_{1y} [(\phi_8 (\phi_4 \psi_{2y} + \psi_2 \phi_{4y}) + \phi_3 \phi_6 \psi_{2y})], \quad (10b)$$

$$\phi_6 \phi_8 \psi_{2x} [(\phi_7 (\phi_2 \psi_{1x} + \psi_1 \phi_{2x}) + \phi_1 \phi_5 \psi_{1x})] = \phi_5 \phi_7 \psi_{1x} [(\phi_8 (\phi_4 \psi_{2x} + \psi_2 \phi_{4x}) + \phi_3 \phi_6 \psi_{2x})], \quad (10c)$$

$$\phi_6 \phi_8 \psi_{2x} [(\phi_7 (\phi_2 \psi_{1y} + \psi_1 \phi_{2y}) + \phi_1 \phi_5 \psi_{1y})] = \phi_5 \phi_7 \psi_{1y} [(\phi_8 (\phi_4 \psi_{2y} + \psi_2 \phi_{4y}) + \phi_3 \phi_6 \psi_{2y})]. \quad (10d)$$

Here, it is to be noted that one can also consider the equation including the terms \dot{x} and \dot{y} , as in Eq. (2), that is the case of coupled mixed Liénard type equation. However, the above set of equations are the determining equations for the functions ϕ_2 and ϕ_4 (see Sec. 3 below). It is clear that considering the linear velocity term to be zero simplifies the above set of determining equations. If the linear velocity term is not zero then it corresponds to mixed Liénard type equation, Eq. (2), which will be discussed in Sec. 9.

Hence, we have factorized Eq. (1) in the form of Eq. (4). The connection between the coefficients of both the equations are given by the relations (7)-(10). With the help of these relations one can obtain the factorized form by inverting the relations.

3. Systematic algorithm for obtaining the factorized form

In order to get the form of the factorized equation (4) we have to solve the relations (7)-(10). Here, in this section, we present a systematic algorithm to obtain the functions ϕ_k' s in (4) by applying appropriate compatibility conditions on them.

Now, inverting the relations (8), one can get the forms of ϕ_1 and ϕ_3 as

$$\phi_1 = \frac{\phi_5 \phi_7}{\phi_2} \left[\frac{g_1 \psi_{1x} + g_2 \psi_{1y}}{\psi_1} \right], \quad \phi_3 = \frac{\phi_6 \phi_8}{\phi_4} \left[\frac{g_1 \psi_{2x} + g_2 \psi_{2y}}{\psi_2} \right]. \quad (11)$$

Multiplying Eq. (10a) with ψ_{1x} and Eq. (10c) with ψ_{1y} and adding, we get

$$\phi_6\phi_8(\phi_7(\psi_1\phi_{2x} + \psi_{1x}\phi_2) + \phi_1\phi_5\psi_{1x})(\psi_{1y}\psi_{2x} - \psi_{1x}\psi_{2y}) = 0. \quad (12)$$

Using ϕ_1 from Eq. (11), we can rewrite the above equation in the form

$$\phi_2\phi_{2x}\psi_1^2 + \phi_2^2\psi_1\psi_{1x} + \phi_5^2\psi_{1x}(g_1\psi_{1x} + g_2\psi_{1y}) = 0. \quad (13)$$

Again multiplying Eq. (10a) with ψ_{2x} and Eq. (10c) with ψ_{2y} and adding, we get

$$\phi_4\phi_{4x}\psi_2^2 + \phi_4^2\psi_2\psi_{2x} + \phi_6^2\psi_{2x}(g_1\psi_{2x} + g_2\psi_{2y}) = 0. \quad (14)$$

Similarly, doing the same for Eqs. (10b) and (10d), we arrive at the following relations,

$$\phi_2\phi_{2y}\psi_1^2 + \phi_2^2\psi_1\psi_{1y} + \phi_5^2\psi_{1y}(g_1\psi_{1x} + g_2\psi_{1y}) = 0, \quad (15a)$$

$$\phi_4\phi_{4y}\psi_2^2 + \phi_4^2\psi_2\psi_{2y} + \phi_6^2\psi_{2y}(g_1\psi_{2x} + g_2\psi_{2y}) = 0. \quad (15b)$$

Note that Eqs. (13)-(15) are effectively Riccati type equations for ϕ_2^2 and ϕ_4^2 . With the help of Eqs. (7a) and (7d) one can get

$$\phi_{5x}\psi_{1x} - (h_1\psi_{1x} + h_4\psi_{1y} - \psi_{1xx})\phi_5 = 0, \quad (16a)$$

$$\phi_{6x}\psi_{2x} - (h_1\psi_{2x} + h_4\psi_{2y} - \psi_{2xx})\phi_6 = 0, \quad (16b)$$

whereas from Eqs. (7b) and (7e) we get

$$\phi_{5y}\psi_{1y} - (h_2\psi_{1x} + h_5\psi_{1y} - \psi_{1yy})\phi_5 = 0, \quad (17a)$$

$$\phi_{6y}\psi_{2y} - (h_2\psi_{2x} + h_5\psi_{2y} - \psi_{2yy})\phi_6 = 0. \quad (17b)$$

Now, from Eqs. (7c) and (7f) one can get a set of PDEs for the functions ϕ_5 and ϕ_6 as

$$\psi_{1y}\phi_{5x} + \psi_{1x}\phi_{5y} + [2\psi_{1xy} - h_3\psi_{1x} - h_6\psi_{1y}]\phi_5 = 0, \quad (18a)$$

$$\psi_{2y}\phi_{6x} + \psi_{2x}\phi_{6y} + [2\psi_{2xy} - h_3\psi_{2x} - h_6\psi_{2y}]\phi_6 = 0. \quad (18b)$$

Substituting the values of ϕ_{5x} , ϕ_{5y} , ϕ_{6x} and ϕ_{6y} from Eqs. (16)-(17) in Eqs. (18), we get a set of PDEs for ψ_1 and ψ_2 as

$$\begin{aligned} \psi_{1y}^2\psi_{1xx} + \psi_{1x}^2\psi_{1yy} - 2\psi_{1x}\psi_{1y}\psi_{1xy} - h_2\psi_{1x}^3 - h_4\psi_{1y}^3 - (h_5 - h_3)\psi_{1x}^2\psi_{1y} \\ - (h_1 - h_6)\psi_{1x}\psi_{1y}^2 = 0, \end{aligned} \quad (19a)$$

$$\begin{aligned} \psi_{2y}^2\psi_{2xx} + \psi_{2x}^2\psi_{2yy} - 2\psi_{2x}\psi_{2y}\psi_{2xy} - h_2\psi_{2x}^3 - h_4\psi_{2y}^3 - (h_5 - h_3)\psi_{2x}^2\psi_{2y} \\ - (h_1 - h_6)\psi_{2x}\psi_{2y}^2 = 0. \end{aligned} \quad (19b)$$

To get the factorized form for Eqs. (1), we need to solve the set of PDEs (19). Once we know the forms of ψ_1 and ψ_2 we can proceed further to obtain the ϕ_k 's. Now substituting the forms of ψ_1 and ψ_2 obtained by solving (19) into Eqs. (16)-(17) one can easily get the values of ϕ_5 and ϕ_6 which on substitution in Eqs. (15) gives the values of ϕ_2 and ϕ_4 . Here, it is to be noted that the structure of the factorized form (4) suggests that one can always define $\tilde{\phi}_1 = \frac{\phi_1}{\phi_7}$ and $\tilde{\phi}_3 = \frac{\phi_3}{\phi_8}$. Now, we can rewrite Eq. (4) in terms of new functions $\tilde{\phi}_1$ and $\tilde{\phi}_3$ as

$$[D - \tilde{\phi}_1(x, y)][\phi_5(x, y)D - \phi_2(x, y)]\psi_1(x, y) = 0, \quad (20a)$$

$$[D - \tilde{\phi}_3(x, y)][\phi_6(x, y)D - \phi_4(x, y)]\psi_2(x, y) = 0. \quad (20b)$$

Then, the forms of $\tilde{\phi}_1$ and $\tilde{\phi}_3$ can be obtained from Eq. (11). Now, we can write the factorized form in terms of $\tilde{\phi}_1$ and $\tilde{\phi}_3$ (vide Eq. (20)). Using the relations $\tilde{\phi}_1 = \frac{\phi_1}{\phi_7}$ and $\tilde{\phi}_3 = \frac{\phi_3}{\phi_8}$ the original factorized form, that is Eq. (4) can be obtained. In this way the factorization of Eq. (1) is complete.

4. Determination of the functions ϕ_k 's

In this section, we discuss the procedure to obtain the form of the functions $\psi_{1,2}$ and ϕ_k 's, $k = 1, 2, \dots, 8$, systematically.

To start with, we integrate Eq. (16a) to obtain the form of ϕ_5 as

$$\phi_5 = \frac{c_1(y)}{\psi_{1x}} e^{\int \left(h_1 + h_4 \frac{\psi_{1x}}{\psi_{1y}} \right) dx}, \quad (21)$$

where $c_1(y)$ is an arbitrary function of y . Similarly, solving Eq. (17a) we get yet another form of ϕ_5 as

$$\phi_5 = \frac{c_2(x)}{\psi_{1y}} e^{\int \left(h_5 + h_2 \frac{\psi_{1x}}{\psi_{1y}} \right) dy}, \quad (22)$$

where $c_2(x)$ is an arbitrary functions of x . The relation between the functions c_1 and c_2 corresponding to Eqs. (21) and (22) can be determined by comparing both the forms of ϕ_5 . In doing so, we get

$$\frac{c_1}{c_2} = \frac{\psi_{1x}}{\psi_{1y}} e^{\int \left(h_5 + h_2 \frac{\psi_{1x}}{\psi_{1y}} \right) dy - \int \left(h_1 + h_4 \frac{\psi_{1x}}{\psi_{1y}} \right) dx}. \quad (23)$$

Hence, the form of ϕ_5 can be obtained from Eqs. (21) or (22) provided c_1 and c_2 satisfy the relation (23). Similarly, the form of ϕ_6 can be obtained by solving Eqs. (16b) and (17b) as

$$\phi_6 = \frac{c_3(y)}{\psi_{2x}} e^{\int \left(h_1 + h_4 \frac{\psi_{2y}}{\psi_{2x}} \right) dx}, \quad (24)$$

and

$$\phi_6 = \frac{c_4(x)}{\psi_{2y}} e^{\int \left(h_5 + h_2 \frac{\psi_{2x}}{\psi_{2y}} \right) dy}, \quad (25)$$

where $c_3(y)$ and $c_4(x)$ are arbitrary functions of x and y , respectively. Here again, c_3 and c_4 are related by

$$\frac{c_3}{c_4} = \frac{\psi_{2x}}{\psi_{2y}} e^{\int \left(h_5 + h_2 \frac{\psi_{2x}}{\psi_{2y}} \right) dy - \int \left(h_1 + h_4 \frac{\psi_{2y}}{\psi_{2x}} \right) dx}. \quad (26)$$

Substituting the form of ϕ_5 in Eqs. (13) and (15a) and solving, we get the form of ϕ_2 as

$$\phi_2 = \frac{\sqrt{2}}{\psi_1} \sqrt{c_5(y) - \int \phi_5^2 \psi_{1x} (g_1 \psi_{1x} + g_2 \psi_{1y}) dx}, \quad (27)$$

and

$$\phi_2 = \frac{\sqrt{2}}{\psi_1} \sqrt{c_6(x) - \int \phi_5^2 \psi_{1y} (g_1 \psi_{1x} + g_2 \psi_{1y}) dy}, \quad (28)$$

where $c_5(y)$ and $c_6(x)$ are arbitrary function of x and y , respectively, which are related by the relation

$$c_5(y) - c_6(x) = \int \phi_5^2 \psi_{1x} (g_1 \psi_{1x} + g_2 \psi_{1y}) dx - \int \phi_5^2 \psi_{1y} (g_1 \psi_{1x} + g_2 \psi_{1y}) dy. \quad (29)$$

Further, from Eqs. (13) and (15a), one can check that the two expressions for ϕ_2 should satisfy the compatibility criterion

$$\frac{\partial}{\partial y} [\phi_5^2 \psi_{1x} (g_1 \psi_{1x} + g_2 \psi_{1y})] = \frac{\partial}{\partial x} [\phi_5^2 \psi_{1y} (g_1 \psi_{1x} + g_2 \psi_{1y})]. \quad (30)$$

Similarly, the form of ϕ_4 can be written with the help of Eqs. (14) and (15b) as

$$\phi_4 = \frac{\sqrt{2}}{\psi_2} \sqrt{c_7(y) - \int \phi_6^2 \psi_{2x} (g_1 \psi_{2x} + g_2 \psi_{2y}) dx}, \quad (31)$$

and

$$\phi_4 = \frac{\sqrt{2}}{\psi_2} \sqrt{c_8(x) - \int \phi_6^2 \psi_{2y} (g_1 \psi_{2x} + g_2 \psi_{2y}) dy}, \quad (32)$$

where the arbitrary functions $c_7(y)$ and $c_8(x)$ are related by the expression

$$c_7(y) - c_8(x) = \int \phi_6^2 \psi_{2x} (g_1 \psi_{2x} + g_2 \psi_{2y}) dx - \int \phi_6^2 \psi_{2y} (g_1 \psi_{2x} + g_2 \psi_{2y}) dy \quad (33)$$

and ϕ_4 should satisfy the compatibility criterion (as may be seen from Eqs. (14) and (15b))

$$\frac{\partial}{\partial y} [\phi_6^2 \psi_{2x} (g_1 \psi_{2x} + g_2 \psi_{2y})] = \frac{\partial}{\partial x} [\phi_6^2 \psi_{2y} (g_1 \psi_{2x} + g_2 \psi_{2y})]. \quad (34)$$

Now, the forms of the functions $\tilde{\phi}_1$ and $\tilde{\phi}_3$ can be obtained from Eqs. (11) as

$$\tilde{\phi}_1 = \frac{c_1(g_1 \psi_{1x} + g_2 \psi_{1y})}{\sqrt{2} \psi_{1x}} \frac{e^{\int (h_1 + h_4 \frac{\psi_{1x}}{\psi_{1y}}) dx}}{\sqrt{c_5(y) - \int \phi_5^2 \psi_{1x} (g_1 \psi_{1x} + g_2 \psi_{1y}) dx}}, \quad (35)$$

and

$$\tilde{\phi}_3 = \frac{c_4(g_1 \psi_{2x} + g_2 \psi_{2y})}{\sqrt{2} \psi_{2x}} \frac{e^{\int (h_1 + h_4 \frac{\psi_{2x}}{\psi_{2y}}) dx}}{\sqrt{c_7(y) - \int \phi_6^2 \psi_{2x} (g_1 \psi_{2x} + g_2 \psi_{2y}) dx}}. \quad (36)$$

Now, we know the forms of ϕ_k 's in terms of the h_i 's, g_j 's and $\psi_{1,2}$ where the only unknown terms are $\psi_{1,2}$. So the problem of factorizing the system under consideration reduces to determining the suitable forms of $\psi_{1,2}$ which will be discussed in the next section.

5. Determination of the functions $\psi_{1,2}$

In this section, we discuss the procedure to identify suitable forms for the functions $\psi_{1,2}$, so that the factorization can be completed. For this purpose, we proceed as follows.

It is always possible to rewrite the given set of coefficients h_i 's in (1) with a common denominator. Hence, we define

$$h_i = \frac{\kappa_i(x, y)}{G(x, y)^p}, \quad i = 1, 2, \dots, 8, \quad (37)$$

where $\kappa_i(x, y)$ and $G(x, y)$ are functions of x and y and p is an arbitrary parameter. Now, to obtain the forms of ψ_1 and ψ_2 we need to solve the set of coupled PDEs (19). For this purpose, we need an ansatz for the functions ψ_1 and ψ_2 as it is very difficult to solve otherwise. One can consider the form of $\psi_{1,2}$ as a rational one where the numerator and denominator are both functions of x and y . Now, as the coefficients h_i 's are also of rational form, we consider the denominator of $\psi_{1,2}$ as $G(x, y)$. Since $\psi_{1,2}$ is in rational form while taking differentiation or integration the form of the denominator remains the same but the power of the denominator decreases or increases by a unit order from that of the initial one. Thus, instead of considering the denominator to be just of the form $G(x, y)$ one can consider a more general form as $G^q(x, y)$, where q is an arbitrary real number. Hence the ansatz for $\psi_{1,2}$ can be considered as

$$\psi_{1,2} = \frac{F_{1,2}(x, y)}{G^q(x, y)}, \quad (38)$$

where $F_{1,2}(x, y)$ are functions to be determined. Substituting the above forms of ψ_1 and h_i 's in Eq. (19a) and simplifying the latter, we get

$$\begin{aligned} G^p \{ & [D_y(F_1 G^q)]^2 [D_x^2(F_1 G^q)] + [D_x(F_1 G^q)]^2 [D_y^2(F_1 G^q)] - 2[D_x(F_1 G^q)][D_y(F_1 G^q)] \\ & \times [D_x D_y(F_1 G^q)] \} - 2q F_1 G^{p+q-2} \{ [D_x(G)]^2 + [D_y(G)]^2 - 2[D_x(G)][D_y(G)] \\ & - G(D_x^2(G) + D_y^2(G) - 2D_x D_y(G)) \} - \kappa_2 [D_x(F_1 G^q)]^3 - \kappa_4 [D_y(F_1 G^q)]^3 \\ & + (\kappa_1 - \kappa_6) [D_x(F_1 G^q)][D_y(F_1 G^q)]^2 - (\kappa_5 - \kappa_3) [D_y(F_1 G^q)][D_x(F_1 G^q)]^2 = 0, \end{aligned} \quad (39)$$

and identically for F_2 , where the Hirota's D -operator is defined as

$$D_x^n (f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1) g(x_2) |_{x_2=x_1=x}. \quad (40)$$

As we know the form of the function G , we need only to determine the forms of the functions $F_{1,2}$. For obtaining admissible forms of $\psi_{1,2}$ we need to choose suitable ansatz for $F_{1,2}$. For example, we can consider $F_{1,2}$ as polynomials in x and y . Substituting the polynomial forms of $F_{1,2}$ into their determining equations (39) and the identical equation for F_2 , and equating the various functions of x and y to zero we get a set of resultant determining equations. Solving the obtained resultant equations we can determine the suitable forms of $F_{1,2}$ which in turn fixes the form of $\psi_{1,2}$. Hence, we can conclude that in principle it is always possible to get the factorized form for Eq. (1) for all the forms of the functions h_i and g_j , provided $F_{1,2}$'s can be found.

6. Example:

In this section, we demonstrate the effectiveness of the procedure discussed in the previous sections by considering an example of this class and then factorizing it properly. We consider the

general form of coupled Mathews-Lakshmanan oscillators [15, 16, 17, 18], that is

$$\ddot{x} - \frac{\lambda x(1 + \lambda y^2)\dot{x}^2 + \lambda x(1 + \lambda x^2)\dot{y}^2 - 2\lambda^2 x^2 y \dot{x}\dot{y} - \alpha_1 x}{1 + \lambda r^2} = 0, \quad (41a)$$

$$\ddot{y} - \frac{\lambda y(1 + \lambda y^2)\dot{x}^2 + \lambda y(1 + \lambda x^2)\dot{y}^2 - 2\lambda^2 x y^2 \dot{x}\dot{y} - \alpha_2 y}{1 + \lambda r^2} = 0, \quad (41b)$$

where $r^2 = x^2 + y^2$ and λ, α_1 and α_2 are arbitrary constants.

6.1. Factorization of Coupled ML oscillator equation

As a first step, we compare Eq. (41) with Eq. (1) to obtain the forms of the functions h_i' and g_j' as

$$\begin{aligned} h_1 &= -\frac{\lambda x(1 + \lambda y^2)}{1 + \lambda r^2}, h_2 = -\frac{\lambda x(1 + \lambda x^2)}{1 + \lambda r^2}, h_3 = \frac{2\lambda^2 x^2 y}{1 + \lambda r^2}, g_1 = \frac{\alpha_1 x}{1 + \lambda r^2}, \\ h_4 &= -\frac{\lambda y(1 + \lambda y^2)}{1 + \lambda r^2}, h_5 = -\frac{\lambda y(1 + \lambda x^2)}{1 + \lambda r^2}, h_6 = \frac{2\lambda^2 x y^2}{1 + \lambda r^2}, g_2 = \frac{\alpha_2 y}{1 + \lambda r^2}. \end{aligned} \quad (42)$$

It is clear from Eq. (42) that all the h_i' have a common denominator. Hence, we can construct suitable forms of $\psi_{1,2}$ with the help of the procedure discussed in the previous sections. For this, we consider the function $F_{1,2}(x, y)$ as polynomial functions in x and y as

$$F_{1,2}(x, y) = a_{1,2}x + b_{1,2}y, \quad (43)$$

where $a_{1,2}$ and $b_{1,2}$ are arbitrary parameters. Now, substituting this form of $F_{1,2}$ in Eq. (39) for F_1 and its counterpart for F_2 and equating the various coefficients of the independent parameters to zero we get a set of algebraic equations in $a_{1,2}$, $b_{1,2}$, p and q . Solving these equations consistently, we get

$$a_{1,2} = \text{arbitrary}, b_{1,2} = \text{arbitrary}, p = 1, q = \frac{1}{2}. \quad (44)$$

Thus, we find that the following forms of $\psi_{1,2}$ are compatible,

$$\psi_1 = \frac{x}{\sqrt{1 + \lambda r^2}}, \psi_2 = \frac{y}{\sqrt{1 + \lambda r^2}}. \quad (45)$$

It is to be noted that the above forms of $\psi_{1,2}$ also satisfy the compatibility criteria given by Eqs. (30) and (34). Hence we can now determine the forms of ϕ_5 and ϕ_6 . For this purpose, we substitute Eq. (45) into Eqs. (21) and (22) and solving them consistently, we get the form of ϕ_5 as

$$\phi_5 = 1 + \lambda r^2, \quad (46)$$

where the relation between $c_1(y)$ and $c_2(x)$ from Eq. (23) turns out to be

$$\frac{c_1}{c_2} = -\frac{1 + \lambda y^2}{\lambda x}. \quad (47)$$

As c_1 is function of y only and c_2 is a function of x only, we can identify $c_1(y) = 1 + \lambda y^2$ and $c_2(x) = -\lambda x$. Similarly, the form of ϕ_6 can be obtained from Eqs. (24) and (25) as

$$\phi_6 = 1 + \lambda r^2, \quad (48)$$

where from Eq. (26) we have $\frac{c_3}{c_4} = -\frac{\lambda y}{1+\lambda x^2}$. Hence, we can fix $c_3(y) = -\lambda y$ and $c_4(x) = 1 + \lambda x^2$ as c_3 is a function of y alone, whereas c_4 is a function of x alone.

Now, to get the form of ϕ_2 and ϕ_4 we first check the compatibility conditions given by Eqs. (30) and (34). Doing so we have to necessarily fix $\alpha_1 = \alpha_2 = \alpha$ in Eq. (42). Now, the function ϕ_2 can be obtained from Eqs. (27) or (28) as

$$\phi_2 = \pm \sqrt{-\alpha}, \quad (49)$$

where the relation between the functions c_5 and c_6 turns out to be $c_5 - c_6 = -\frac{\alpha}{2\lambda}$. The relation between c_5 and c_6 suggest various possibilities for their forms. However, we need to find c_5 and c_6 such that they are consistent with Eqs. (27) and (28). For example, one can consider three simplest forms as (i) $c_5 = -\frac{\alpha}{2\lambda}$ and $c_6 = 0$, (ii) $c_5 = 0$ and $c_6 = -\frac{\alpha}{2\lambda}$ and (iii) $c_5 = -\frac{\alpha}{4\lambda}$ and $c_6 = \frac{\alpha}{4\lambda}$. One can check that only case (i) is consistent with Eqs. (27) and (28). Similarly, we can get the form of ϕ_4 as

$$\phi_4 = \pm \sqrt{-\alpha}, \quad (50)$$

where $c_7(y) = 0$ and $c_8(x) = -\frac{\alpha}{2\lambda}$. Here also we need to fix the forms of c_7 and c_8 so that they are consistent with Eqs. (31) and (32). One can check that the other obvious choices, that is $c_7(y) = -\frac{\alpha}{2\lambda}$ and $c_8(x) = 0$ and $c_5 = -\frac{\alpha}{4\lambda}$ and $c_6 = \frac{\alpha}{4\lambda}$ are not consistent with Eqs. (31) and (32).

Finally, the forms of $\tilde{\phi}_1$ and $\tilde{\phi}_3$ can be obtained with the help of Eqs. (35) and (36) as

$$\tilde{\phi}_1 = \tilde{\phi}_3 = \mp \frac{\sqrt{-\alpha}}{1 + \lambda r^2}. \quad (51)$$

Now, we know the forms of all the functions ϕ_k 's, $k = 1, 2, \dots, 8$, which completes the factorization of the coupled ML oscillator. The factorized form can be then written as

$$\left[(1 + \lambda r^2)D \pm \sqrt{-\alpha} \right] \left[(1 + \lambda r^2)D \mp \sqrt{-\alpha} \right] \frac{x}{\sqrt{1 + \lambda r^2}} = 0, \quad (52a)$$

$$\left[(1 + \lambda r^2)D \pm \sqrt{-\alpha} \right] \left[(1 + \lambda r^2)D \mp \sqrt{-\alpha} \right] \frac{y}{\sqrt{1 + \lambda r^2}} = 0. \quad (52b)$$

6.2. Integrability of coupled ML oscillator equation

The above factorized form can be used to obtain the general solution of the coupled ML oscillator. For this purpose we can write Eq. (52a) with the choice of both the signs as

$$\left[(1 + \lambda r^2)D + \sqrt{-\alpha} \right] \left[(1 + \lambda r^2)D - \sqrt{-\alpha} \right] \frac{x}{\sqrt{1 + \lambda r^2}} = 0, \quad (53a)$$

$$\left[(1 + \lambda r^2)D - \sqrt{-\alpha} \right] \left[(1 + \lambda r^2)D + \sqrt{-\alpha} \right] \frac{x}{\sqrt{1 + \lambda r^2}} = 0. \quad (53b)$$

Considering $\tilde{D} = (1 + \lambda r^2)D$, the above Eqs. (53) can be written in the form of the harmonic oscillator like equation,

$$\left[\tilde{D} + \sqrt{-\alpha} \right] \left[\tilde{D} - \sqrt{-\alpha} \right] \frac{x}{\sqrt{1 + \lambda r^2}} = 0, \quad (54a)$$

$$\left[\tilde{D} - \sqrt{-\alpha} \right] \left[\tilde{D} + \sqrt{-\alpha} \right] \frac{x}{\sqrt{1 + \lambda r^2}} = 0. \quad (54b)$$

One can always rewrite Eqs. (54) in the form

$$\frac{\tilde{D}\left[\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] - \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}\right]}{\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] - \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}} + \frac{\tilde{D}\left[\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] + \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}\right]}{\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] + \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}} = 0. \quad (55)$$

Integrating Eq. (55) once, we arrive at

$$I_1 = \left(\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] + \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}\right)\left(\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right] - \sqrt{-\alpha}\frac{x}{\sqrt{1+\lambda r^2}}\right), \quad (56)$$

where I_1 is an integration constant. Now, (56) can be rewritten as

$$I_1 = \left(\tilde{D}\left[\frac{x}{\sqrt{1+\lambda r^2}}\right]\right)^2 + \alpha\left(\frac{x}{\sqrt{1+\lambda r^2}}\right)^2. \quad (57)$$

Similarly, one can write from Eq. (52b) as

$$I_2 = \left(\tilde{D}\left[\frac{y}{\sqrt{1+\lambda r^2}}\right]\right)^2 + \alpha\left(\frac{y}{\sqrt{1+\lambda r^2}}\right)^2, \quad (58)$$

where I_2 is a constant of integration. Now, substituting $\tilde{D} = (1 + \lambda r^2)D$ and simplifying Eqs. (57) and (58), we get

$$\left(\dot{x}(1 + \lambda y^2) - \lambda xy\dot{y}\right)^2 + \alpha x^2 = I_1(1 + \lambda r^2), \quad (59)$$

and

$$\left(\dot{y}(1 + \lambda x^2) - \lambda xy\dot{x}\right)^2 + \alpha y^2 = I_2(1 + \lambda r^2). \quad (60)$$

To deduce the third integration constant we use the first integral of harmonic oscillator equation

$$I_3 = \psi_2 \tilde{D}\psi_1 - \psi_1 \tilde{D}\psi_2, \quad (61)$$

where $\tilde{D} = (1 + \lambda r^2)\frac{d}{dt}$. Using the forms of ψ_1, ψ_2 and \tilde{D} in Eq. (61) and simplifying, we get the form of the integration constant \tilde{I}_2 as

$$I_3 = \dot{x}y - x\dot{y}. \quad (62)$$

Adding Eqs. (59) and (60) and simplifying, we arrive at

$$\tilde{I}_1 = \frac{\alpha r^2 + \dot{x}^2 \left((1 + \lambda y^2)^2 + \lambda^2 x^2 y^2 \right) + \dot{y}^2 \left((1 + \lambda x^2)^2 + \lambda^2 x^2 y^2 \right) - 2\lambda xy\dot{x}\dot{y}(2 + \lambda r^2)}{1 + \lambda r^2}, \quad (63)$$

where $\tilde{I}_1 = I_1 + I_2$ is a new integration constant. Once we know the form of the two integrals of motion \tilde{I}_1 and I_3 , we can deduce the linearizing transformations with the help of the procedure

discussed by Chandrasekar *et al.* [17]. For this purpose, we consider the first integrals \tilde{I}_1 and I_3 . Now, rewriting the first integrals in the form

$$\begin{aligned}\tilde{I}_1 &= \frac{\alpha r^2 + \dot{x}^2 \left((1 + \lambda y^2)^2 + \lambda^2 x^2 y^2 \right) + \dot{y}^2 \left((1 + \lambda x^2)^2 + \lambda^2 x^2 y^2 \right) - 2\lambda xy \dot{x} \dot{y} (2 + \lambda r^2)}{2\lambda(x\dot{x} + y\dot{y})} \\ &\quad \times \frac{d}{dt} \log(1 + \lambda r^2) = \frac{dw_1}{dz_1},\end{aligned}\quad (64)$$

$$I_3 = y^2 \frac{d}{dt} \left(\frac{x}{y} \right) = \frac{dw_2}{dz_2}, \quad (65)$$

we identify the following set of linearizing transformations

$$\begin{aligned}w_1 &= \log(1 + \lambda r^2), & w_2 &= \frac{x}{y}, \\ z_1 &= \int \frac{2\lambda(x\dot{x} + y\dot{y}) dt}{\alpha r^2 + \dot{x}^2 \left((1 + \lambda y^2)^2 + \lambda^2 x^2 y^2 \right) + \dot{y}^2 \left((1 + \lambda x^2)^2 + \lambda^2 x^2 y^2 \right) - 2\lambda xy \dot{x} \dot{y} (2 + \lambda r^2)}, \\ z_2 &= \int \frac{dt}{y^2}.\end{aligned}\quad (66)$$

Rewriting the first integrals \tilde{I}_1 and I_3 in the integral form and identifying them in terms of the new variables, we get $w_1 = \tilde{I}_1 z_1$ and $w_2 = I_3 z_2$. From this one can get the relation between the variables x and y with z_1 and z_2 , respectively, (the integration constant is fixed to be zero without loss of generality) as

$$1 + \lambda r^2 = e^{\tilde{I}_1 z_1} \quad \text{and} \quad x = I_3 z_2 y. \quad (67)$$

Making use of Eqs. (63), (62) and (66), we can write

$$dz_1 = \frac{2\lambda \sqrt{(\tilde{I}_1 - I_3^2 (1 + \lambda r^2))(1 + \lambda r^2) - \alpha r^4}}{I_1 (1 + \lambda r^2)} dt. \quad (68)$$

Using the result $1 + \lambda r^2 = e^{\tilde{I}_1 z_1}$ the above expression can be rewritten as

$$dz_1 = \frac{2\lambda}{\tilde{I}_1} \sqrt{\left(\tilde{I}_1 + \frac{2\alpha}{\lambda^2} \right) e^{-\tilde{I}_1 z_1} - \left(I_3^2 + \frac{\alpha}{\lambda^2} \right) - \frac{\alpha}{\lambda^2} e^{-2\tilde{I}_1 z_1}} dt. \quad (69)$$

To get the third integration constant we integrate the above equation. Doing this, we get

$$I_4 - t = \frac{1}{2\lambda \sqrt{I_3^2 + \frac{\alpha}{\lambda^2}}} \tan^{-1} \left[\frac{2 \left(I_3^2 + \frac{\alpha}{\lambda^2} \right) - \left(\tilde{I}_1 + \frac{2\alpha}{\lambda^2} \right) e^{-\tilde{I}_1 z_1}}{2 \sqrt{I_3^2 + \frac{\alpha}{\lambda^2}} \sqrt{\left(\tilde{I}_1 + \frac{2\alpha}{\lambda^2} \right) e^{-\tilde{I}_1 z_1} - \left(I_3^2 + \frac{\alpha}{\lambda^2} \right) - \frac{\alpha}{\lambda^2} e^{-2\tilde{I}_1 z_1}}} \right], \quad (70)$$

where I_4 is the fourth integral of motion. Now making use these four integrals of motion, namely (59), (60), (62) and (70), the general solution can be straightforwardly constructed. The resultant solution also agrees with Eq. (5.40) of Chandrasekar *et al.* [17].

7. Isochronous condition

In the previous sections, we paid our attention to get the factorized form of Eq. (1) systematically. Now, in this part of the paper we are interested in identifying the form of the equation belonging to Eq. (1) which exhibits isochronous properties. For this purpose, we transform our system (Eq. (1)) into a set of uncoupled simple harmonic oscillator equations as the latter ones are prototypes of isochronous systems.

Let us consider a system uncoupled harmonic oscillator equations of the form

$$\ddot{\psi}_1 + \omega_1 \psi_1 = 0, \quad (71a)$$

$$\ddot{\psi}_2 + \omega_2 \psi_2 = 0. \quad (71b)$$

Eqs. (71) can be rewritten in the form of factorized equations as

$$[D \pm \sqrt{-\omega_1}][D \mp \sqrt{-\omega_1}]\psi_1 = 0, \quad (72a)$$

$$[D \pm \sqrt{-\omega_2}][D \mp \sqrt{-\omega_2}]\psi_2 = 0, \quad (72b)$$

where ω_1 and ω_2 are constants and ψ_1 and ψ_2 are eigen functions. If system (4) exhibits isochronous property then it can be obtained from Eq. (72) with the help of a suitable transformation. Hence, choosing the form of the functions ψ_i appropriately one can transform Eq. (72) to Eq. (4). Then one can easily identify the forms of the functions in (4) as $\phi_1 = \phi_2 = \sqrt{-\omega_1}$, $\phi_3 = \phi_4 = \sqrt{-\omega_2}$ and $\phi_5 = \phi_6 = \phi_7 = \phi_8 = 1$. With the help of these functions the determining Eqs. (15)-(18) can be simplified. The resultant equations for the functions ψ_1 turn out to be

$$g_1\psi_{1x} + g_2\psi_{1y} - \omega_1\psi_1 = 0, \quad (73a)$$

$$h_1\psi_{1x} + h_4\psi_{1y} - \psi_{1xx} = 0, \quad (73b)$$

$$h_2\psi_{1x} + h_5\psi_{1y} - \psi_{1yy} = 0, \quad (73c)$$

$$h_3\psi_{1x} + h_6\psi_{1y} - 2\psi_{1xy} = 0, \quad (73d)$$

where $\psi_{1x} \neq 0$ and $\psi_{1y} \neq 0$. Now, with the help of the first two relations of Eq. (73) one can equate the values of ψ_{1xx} . Then making use of the other relations, we arrive at

$$\psi_{1x}[g_1h_1 + g_{1x} + \frac{1}{2}g_2h_3 - \omega_1] + \psi_{1y}[g_1h_4 + g_{2x} + \frac{1}{2}g_2h_6] = 0. \quad (74)$$

Again, equating the value of ψ_{1yy} from first and third relations of Eqs. (73), we get

$$\psi_{1x}[g_2h_2 + g_{1y} + \frac{1}{2}g_1h_3] + \psi_{1y}[g_2h_5 + g_{2y} + \frac{1}{2}g_1h_6 - \omega_1] = 0. \quad (75)$$

For nontrivial solutions for ψ_{1x} and ψ_{1y} to exist, from Eqs. (74) and (75) we require that

$$\begin{aligned} [g_1h_1 + g_{1x} + \frac{1}{2}g_2h_3 - \omega_1][g_2h_5 + g_{2y} + \frac{1}{2}g_1h_6 - \omega_1] &= [g_1h_4 + g_{2x} + \frac{1}{2}g_2h_6] \\ &\times [g_2h_2 + g_{1y} + \frac{1}{2}g_1h_3]. \end{aligned} \quad (76)$$

Similarly, the modified determining equations for the function ψ_2 are

$$g_1\psi_{2x} + g_2\psi_{2y} - \omega_2\psi_2 = 0, \quad (77a)$$

$$h_1\psi_{2x} + h_4\psi_{2y} - \psi_{2xx} = 0, \quad (77b)$$

$$h_2\psi_{2x} + h_5\psi_{2y} - \psi_{2yy} = 0, \quad (77c)$$

$$h_3\psi_{2x} + h_6\psi_{2y} - 2\psi_{2xy} = 0, \quad (77d)$$

where $\psi_{2x} \neq 0$ and $\psi_{2y} \neq 0$. Following the same procedure for ψ_2 as was done for ψ_1 , we arrive at the relation

$$\begin{aligned} [g_1 h_1 + g_{1x} + \frac{1}{2} g_2 h_3 - \omega_2][g_2 h_5 + g_{2y} + \frac{1}{2} g_1 h_6 - \omega_2] &= [g_1 h_4 + g_{2x} + \frac{1}{2} g_2 h_6] \\ &\times [g_2 h_2 + g_{1y} + \frac{1}{2} g_1 h_3]. \end{aligned} \quad (78)$$

Equating the right hand sides of Eqs. (76) and (78) and simplifying, we get

$$g_1 h_1 + g_{1x} + \frac{1}{2} g_2 h_3 + g_2 h_5 + g_{2y} + \frac{1}{2} g_1 h_6 - \omega_1 - \omega_2 = 0, \quad (79)$$

provided $\omega_1 \neq \omega_2$.

Eq. (79) along with Eq. (76) or (78) can be used to identify the isochronous equations belonging to the general Eq. (4) by imposing these conditions on the form of the functions h_i and g_j .

Limiting Case: It is to be noted that for the scalar quadratic Liénard type equation [10]

$$\ddot{x} + h(x)\dot{x}^2 + g(x) = 0, \quad (80)$$

where $h(x)$ and $g(x)$ are arbitrary functions of x only, the isochronicity condition from Eqs. (76) and (78) turns out to be

$$g_x + hg = \omega_1, \quad (81)$$

which is exactly the same as has been proved by many authors [19, 20, 21, 22, 23].

8. Example of isochronicity

In this section, we consider a physically interesting example and show that it satisfies the isochronous condition given in the previous section and exhibits amplitude independent periodic solutions. We also find their general solutions.

Let us consider a second order ODE of the form [17]

$$\ddot{x} + \frac{(\dot{x}y - \dot{y}x)^2}{2xy(x-y)} + \omega x = 0, \quad \ddot{y} - \frac{(\dot{x}y - \dot{y}x)^2}{2xy(x-y)} + \omega y = 0. \quad (82)$$

Identifying the forms of the functions h_i and g_j as

$$\begin{aligned} h_1 &= \frac{y}{2x(x-y)}, & h_2 &= \frac{x}{2y(x-y)}, & h_3 &= -\frac{1}{x-y}, & g_1 &= \omega x, \\ h_4 &= -\frac{y}{2x(x-y)}, & h_5 &= -\frac{x}{2y(x-y)}, & h_6 &= \frac{1}{x-y}, & g_2 &= \omega y. \end{aligned} \quad (83)$$

One can check that the above forms of h_i and g_i satisfy the isochronicity condition given by Eqs. (76) and (78). Hence, Eq. (82) can be transformed to the system of coupled simple harmonic oscillator equations with appropriate transformation. The system of coupled simple harmonic oscillator equations can be written as

$$\begin{aligned} [D + \sqrt{-\omega}][D - \sqrt{-\omega}](x+y) &= 0, \\ [D + \sqrt{-\omega}][D - \sqrt{-\omega}]\sqrt{xy} &= 0, \end{aligned} \quad (84)$$

where the form of the functions ψ_1 and ψ_2 are obtained by solving Eqs. (73) and (77) consistently. One may check that expanding the above set of equations one can get the system (82) under consideration. It means that the transformation $\psi_1 = x + y$ and $\psi_2 = \sqrt{xy}$ leads the simple harmonic oscillator equation to the desired equation. Hence, the solution of the desired equation can be obtained from the solution of the simple harmonic oscillator equation just by inverting the relation. The solution of simple harmonic oscillator is

$$\psi_1 = A \sin(\omega t + \delta_1), \quad \psi_2 = B \sin(\omega t + \delta_2). \quad (85)$$

Substituting $\psi_1 = x + y$ and $\psi_2 = \sqrt{xy}$ in the above equation and solving, we get the solution of Eq. (82) as

$$x(t) = \frac{1}{2} \left(A \sin(\omega t + \delta_1) \pm \sqrt{A^2 \sin^2(\omega t + \delta_1) - 4B^2 \sin^2(\omega t + \delta_2)} \right), \quad (86)$$

$$y(t) = \frac{1}{2} \left(A \sin(\omega t + \delta_1) \mp \sqrt{A^2 \sin^2(\omega t + \delta_1) - 4B^2 \sin^2(\omega t + \delta_2)} \right). \quad (87)$$

where A, B, δ_1 and δ_2 are integration constants.

9. The case of mixed Liénard type equation

In this section, we consider the addition of a linear velocity term in addition to the quadratic velocity term in Eq. (1) to get an overview of a more general class of equations.

Including the linear velocity term, Eq. (1) can be written as

$$\ddot{x} + h_1(x, y)\dot{x}^2 + h_2(x, y)\dot{y}^2 + h_3(x, y)\dot{x}\dot{y} + f_1(x, y)\dot{x} + f_2(x, y)\dot{y} + g_1(x, y) = 0, \quad (88a)$$

$$\ddot{y} + h_4(x, y)\dot{x}^2 + h_5(x, y)\dot{y}^2 + h_6(x, y)\dot{x}\dot{y} + f_3(x, y)\dot{x} + f_4(x, y)\dot{y} + g_2(x, y) = 0. \quad (88b)$$

Inclusion of the linear velocity terms will not change the form of h_i 's and g_j 's and hence the determining equations for $\psi_{1,2}$, ϕ_5 and ϕ_6 remain unchanged. The only difference is that now the coefficients of \dot{x} and \dot{y} will not be zero but will be given by the functions f_l 's, $l = 1, 2, 3, 4$. Now, comparing the above equation with (4) we get the forms of the functions f_l 's, $l = 1, 2, 3, 4$, as

$$f_1 = \frac{1}{\delta} [-\phi_6 \phi_8 \psi_{2y} (\phi_7 (\psi_1 \phi_{2x} + \psi_{1x} \phi_2) + \phi_1 \phi_5 \psi_{1x}) + \phi_5 \phi_7 \psi_{1y} (\phi_8 (\psi_2 \phi_{4x} + \psi_{2x} \phi_4) + \phi_3 \phi_6 \psi_{2x})], \quad (89a)$$

$$f_2 = \frac{1}{\delta} [-\phi_6 \phi_8 \psi_{2y} (\phi_7 (\psi_1 \phi_{2y} + \psi_{1y} \phi_2) + \phi_1 \phi_5 \psi_{1y}) + \phi_5 \phi_7 \psi_{1y} (\phi_8 (\psi_2 \phi_{4y} + \psi_{2y} \phi_4) + \phi_3 \phi_6 \psi_{2y})], \quad (89b)$$

$$f_3 = \frac{1}{\delta} [\phi_6 \phi_8 \psi_{2x} (\phi_7 (\psi_1 \phi_{2x} + \psi_{1x} \phi_2) + \phi_1 \phi_5 \psi_{1x}) - \phi_5 \phi_7 \psi_{1x} (\phi_8 (\psi_2 \phi_{4x} + \psi_{2x} \phi_4) + \phi_3 \phi_6 \psi_{2x})], \quad (89c)$$

$$f_4 = \frac{1}{\delta} [\phi_6 \phi_8 \psi_{2x} (\phi_7 (\psi_1 \phi_{2y} + \psi_{1y} \phi_2) + \phi_1 \phi_5 \psi_{1y}) - \phi_5 \phi_7 \psi_{1x} (\phi_8 (\psi_2 \phi_{4y} + \psi_{2y} \phi_4) + \phi_3 \phi_6 \psi_{2y})]. \quad (89d)$$

In the case of Eq. (1) the left hand sides of Eqs. (89) are zero. Hence, we can determine ϕ_2 and ϕ_4 easily. To get the form of the functions ϕ_2 and ϕ_4 in the present case we proceed in the following manner.

Multiplying (89a) by ψ_{1x} and (89c) by ψ_{1y} and adding, we get

$$\phi_7(\phi_{2x}\psi_1 + \phi_2\psi_{1x}) = -\phi_5\phi_7(\psi_{1x}f_1 + \psi_{1y}f_3) - \phi_1\phi_5\psi_{1x}. \quad (90)$$

Now, multiplying the above equation by $\psi_1\phi_2$ and using (11), we arrive at the following relation

$$\frac{\partial}{\partial x} \left[\frac{\phi_2^2\psi_1^2}{2} \right] = -(\psi_{1x}g_1 + \psi_{1y}g_2)\phi_5^2\psi_{1x} - (\psi_{1x}f_1 + \psi_{1y}f_3)\phi_2\phi_5\psi_1. \quad (91)$$

Again, multiplying (89a) by ψ_{2x} and (89c) by ψ_{2y} and simplifying in the same manner as we have done above, we find

$$\frac{\partial}{\partial x} \left[\frac{\phi_4^2\psi_2^2}{2} \right] = -(\psi_{2x}g_1 + \psi_{2y}g_2)\phi_6^2\psi_{2x} - (\psi_{2x}f_1 + \psi_{2y}f_3)\phi_4\phi_6\psi_2. \quad (92)$$

Proceeding in the same way for Eqs. (89b) and (89d), we arrive at the following relations

$$\frac{\partial}{\partial y} \left[\frac{\phi_2^2\psi_1^2}{2} \right] = -(\psi_{1x}g_1 + \psi_{1y}g_2)\phi_5^2\psi_{1y} - (\psi_{1x}f_2 + \psi_{1y}f_4)\phi_2\phi_5\psi_1, \quad (93)$$

$$\frac{\partial}{\partial x} \left[\frac{\phi_4^2\psi_2^2}{2} \right] = -(\psi_{2x}g_1 + \psi_{2y}g_2)\phi_6^2\psi_{2y} - (\psi_{2x}f_2 + \psi_{2y}f_4)\phi_4\phi_6\psi_2. \quad (94)$$

Using the compatibility condition of (91) with (92) and of (93) with (94), we can write

$$\begin{aligned} \frac{\partial}{\partial y} \left[(\psi_{1x}g_1 + \psi_{1y}g_2)\phi_5^2\psi_{1x} + (\psi_{1x}f_1 + \psi_{1y}f_3)\phi_2\phi_5\psi_1 \right] \\ = \frac{\partial}{\partial x} \left[(\psi_{1x}g_1 + \psi_{1y}g_2)\phi_5^2\psi_{1y} + (\psi_{1x}f_2 + \psi_{1y}f_4)\phi_2\phi_5\psi_1 \right], \end{aligned} \quad (95a)$$

$$\begin{aligned} \frac{\partial}{\partial y} \left[(\psi_{2x}g_1 + \psi_{2y}g_2)\phi_6^2\psi_{2x} + (\psi_{2x}f_1 + \psi_{2y}f_3)\phi_4\phi_6\psi_2 \right] \\ = \frac{\partial}{\partial x} \left[(\psi_{2x}g_1 + \psi_{2y}g_2)\phi_6^2\psi_{2y} + (\psi_{2x}f_2 + \psi_{2y}f_4)\phi_4\phi_6\psi_2 \right]. \end{aligned} \quad (95b)$$

If we define

$$\hat{g}_1 = \psi_{1x}g_1 + \psi_{1y}g_2, \quad \hat{g}_2 = \psi_{2x}g_1 + \psi_{2y}g_2, \quad (96a)$$

$$\hat{f}_1 = \psi_{1x}f_1 + \psi_{1y}f_3, \quad \hat{f}_2 = \psi_{1x}f_2 + \psi_{1y}f_4, \quad (96b)$$

$$\hat{f}_3 = \psi_{2x}f_1 + \psi_{2y}f_3, \quad \hat{f}_4 = \psi_{2x}f_2 + \psi_{2y}f_4, \quad (96c)$$

then Eqs. (95) can be written as

$$\begin{aligned} (\hat{f}_2\phi_{2x} - \hat{f}_1\phi_{2y})\phi_5\psi_1 + [(\hat{f}_2x - \hat{f}_1y)\phi_5\psi_1 + \hat{f}_2(\phi_{5x}\psi_1 + \phi_5\psi_{1x}) - \hat{f}_1(\phi_{5y}\psi_1 + \phi_5\psi_{1y})]\phi_2 \\ + \phi_5^2(\hat{g}_{1x}\psi_{1y} - \hat{g}_{1y}\psi_{1x}) + 2\hat{g}_1\phi_5(\phi_{5x}\psi_{1y} - \phi_{5y}\psi_{1x}) = 0, \end{aligned} \quad (97)$$

$$\begin{aligned} (\hat{f}_4\phi_{4x} - \hat{f}_3\phi_{4y})\phi_6\psi_2 + [(\hat{f}_4x - \hat{f}_3y)\phi_6\psi_2 + \hat{f}_4(\phi_{6x}\psi_2 + \phi_6\psi_{2x}) - \hat{f}_3(\phi_{6y}\psi_2 + \phi_6\psi_{2y})]\phi_4 \\ + \phi_6^2(\hat{g}_{2x}\psi_{2y} - \hat{g}_{2y}\psi_{2x}) + 2\hat{g}_2\phi_6(\phi_{6x}\psi_{2y} - \phi_{6y}\psi_{2x}) = 0. \end{aligned} \quad (98)$$

The above set equations are the determining equations for ϕ_2 and ϕ_4 . As mentioned, the determining equations for $\psi_{1,2}$ and ϕ_5 and ϕ_6 are the same as in the case of Eq. (1) and for $\psi_{1,2}$ it is given by (19), and ϕ_5 and ϕ_6 are given by Eqs. (16)-(17). Hence, proceeding in the same way as discussed in Secs. 4 and 5 one can determine the forms of the function $\psi_{1,2}$, ϕ_5 and ϕ_6 . Substituting $\psi_{1,2}$, ϕ_5 and ϕ_6 in Eqs. (95) and solving the resultant PDEs the functions ϕ_2 and ϕ_4 can be obtained. With the help of the known forms in Eq. (11) the forms of ϕ_1 and ϕ_3 can be fixed in terms of ϕ_7 and ϕ_8 .

10. Conclusion

In this paper, we have developed a systematic and self contained procedure which enables us to analyse the factorization of a rather general class of coupled quadratic Liénard type equations in terms of first order differential operators. In this way, we have shown that the factorized form for the given equation can be obtained in a systematic and simple way. This reduces the problem of finding the solution of the equations belonging to this class to the problem of solving a set of first order differential equations. To demonstrate the effectiveness of this procedure we considered coupled ML oscillator equation and factorized it systematically. In addition to this, we have also considered the isochronous properties of this equation and deduced the isochronicity condition for it. With the help of this condition one can identify the forms of Eq. (4) exhibiting isochronous property. An example of physical interest is also discussed. Finally, we have also extended the procedure to the case of coupled mixed type of Liénard equations by including linear velocity terms in addition to the quadratic velocity terms.

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Appendix A. Factorization of scalar case corresponding to Eq. (1)

In this appendix, we consider the scalar case corresponding to Eq. (1), that is

$$\ddot{x} + h(x)\dot{x}^2 + g(x) = 0, \quad (\text{A.1})$$

where $h(x)$ and $g(x)$ are functions of x , and discuss how to get the factorized form for Eq. (A.1). We will also consider a specific example belonging to Eq. (A.1).

Appendix A.1. Factorization of Eq. (A.1)

To start with, let us assume that Eq. (A.1) can be factorized in the form

$$[\phi_4(x)D - \phi_3(x)][\phi_2(x)D - \phi_1(x)]\psi(x) = 0, \quad (\text{A.2})$$

where ϕ_k 's, $k = 1, 2, 3, 4$, and $\psi(x)$ are unknown functions of x to be determined. It is to be noted that one can always absorb ϕ_4 in ϕ_3 by redefining the functions. Hence, we define $\tilde{\phi}_3 = \frac{\phi_3}{\phi_4}$. Under this definition Eq. (A.2) can be written as

$$[D - \tilde{\phi}_3(x)][\phi_2(x)D - \phi_1(x)]\psi(x) = 0. \quad (\text{A.3})$$

Now, expanding Eq. (A.3) and comparing the latter with the coefficients of various powers of \dot{x} of Eq. (A.1), we get

$$(\phi_2\psi_x)_x - h\phi_2\psi_x = 0, \quad (\text{A.4})$$

$$\phi_1\tilde{\phi}_3\psi - \phi_2\psi_x g = 0, \quad (\text{A.5})$$

and

$$(\phi_1 \psi)_x + \phi_2 \tilde{\phi}_3 \psi_x = 0. \quad (\text{A.6})$$

In order to get the factorized form we need to identify the unknown functions, that is $\phi_1, \phi_2, \tilde{\phi}_3$ and ψ . To deduce suitable forms for these unknowns we need to solve Eqs. (A.4)-(A.6) consistently.

Solving Eq. (A.4), we get

$$\phi_2 \psi_x = c_1 e^{\int h dx}, \quad (\text{A.7})$$

where c_1 is an arbitrary constant. Using Eq. (A.7) in Eq. (A.5), we find

$$\phi_1 \tilde{\phi}_3 \psi = g c_1 e^{\int h dx}. \quad (\text{A.8})$$

Similarly, using (A.7) in (A.6), we get

$$\tilde{\phi}_3 e^{\int h dx} = -(\phi_1 \psi)_x. \quad (\text{A.9})$$

Using Eq. (A.8) in (A.9), we get

$$\phi_1 \psi (\phi_1 \psi)_x = -g c_1^2 e^{2 \int h dx}. \quad (\text{A.10})$$

Solving above equation, we get

$$\phi_1 \psi = \sqrt{c_2 - 2c_1^2 \int g e^{2 \int h dx} dx}, \quad (\text{A.11})$$

where c_2 is a constant. Hence, Eqs. (A.7), (A.9) and (A.11) can be used to identify the suitable form of the unknown functions. Now, it is clear from Eqs. (A.3) and (A.11) that we have freedom to choose the form of the function ϕ_1 in order to obtain the factorized form (A.3). Hence, we consider ϕ_1 as a function of x , say $M(x)$. Then from Eq. (A.11), the form of ψ can be written as

$$\psi = \frac{\sqrt{c_2 - 2c_1^2 \int g e^{2 \int h dx} dx}}{M}. \quad (\text{A.12})$$

With the above form of ψ , the form of ϕ_2 can be written from Eq. (A.7) as

$$\phi_2 = -\frac{c_1 M^2 e^{\int h dx} \sqrt{c_2 - 2c_1^2 \int e^{2 \int h dx} g dx}}{g M c_1^2 e^{2 \int h dx} + (c_2 - 2c_1^2 \int e^{2 \int h dx} g dx) M'}, \quad (\text{A.13})$$

where ' ' denotes differentiation with respect to x . Using (A.12) in (A.8) the form of $\tilde{\phi}_3$ turn out to be

$$\tilde{\phi}_3 = \frac{c_1 g e^{\int h dx}}{\sqrt{c_2 - 2c_1^2 \int g e^{2 \int h dx} dx}}. \quad (\text{A.14})$$

These above form of the functions ψ , ϕ_2 and $\tilde{\phi}_3$ given by Eqs. (A.12), (A.13) and (A.14) and $\phi_1 = M$ satisfies Eq. (A.1). It means that even for the arbitrary form of ϕ_1 the above forms of the functions ψ , ϕ_2 and $\tilde{\phi}_3$ give the suitable factorized form for Eq. (A.1). Hence, we can consider ϕ_1 as a constant. To illustrate this procedure, in the following, we consider an example of physical and mathematical interest.

As an example, we consider the ML oscillator equation [15, 16]

$$\ddot{x} - \frac{\lambda x}{1 + \lambda x^2} \dot{x}^2 + \frac{\omega x}{1 + \lambda x^2} = 0, \quad (\text{A.15})$$

where λ and ω are arbitrary parameters. Comparing above equation with Eq. (A.1), we get

$$h = -\frac{\lambda x}{1 + \lambda x^2}, \quad g = \frac{\omega x}{1 + \lambda x^2}. \quad (\text{A.16})$$

If, Eq. (A.16) can be written in factorized form as Eq. (A.3), then we can write the determining equations for the unknowns with the help of Eqs. (A.4)-(A.6). To get the factorized form we consider ϕ_1 as a constant, say $\sqrt{-\omega}$. Then solving Eq. (A.11) for ψ , we get

$$\psi = \frac{x}{\sqrt{1 + \lambda x^2}}, \quad (\text{A.17})$$

where $c_1 = 1$ and $c_2 = -\frac{\omega}{\lambda}$. Now, ϕ_2 can be obtained by solving Eq. (A.7) as

$$\phi_2 = 1 + \lambda x^2 \quad (\text{A.18})$$

and $\tilde{\phi}_3$ can be obtained from (A.8) as

$$\tilde{\phi}_3 = -\frac{\sqrt{-\omega}}{1 + \lambda x^2}. \quad (\text{A.19})$$

With the help of the obtained forms of the unknown functions the factorized form for ML oscillator can be written as

$$\left[(1 + \lambda x^2)D \pm \sqrt{-\omega}\right] \left[(1 + \lambda x^2)D \mp \sqrt{-\omega}\right] \frac{x}{\sqrt{1 + \lambda x^2}} = 0, \quad (\text{A.20})$$

where we have used the relation $\tilde{\phi}_3 = \frac{\phi_3}{\phi_4}$ and $\phi_4 = 1 + \lambda x^2$.

To prove the integrability of Eq. (A.15), we proceed in the same way as was done in Sec. 6.2. Hence, we can write Eq. (A.20) as

$$\left[\tilde{D} + \sqrt{-\omega}\right] \left[\tilde{D} - \sqrt{-\omega}\right] \frac{x}{\sqrt{1 + \lambda x^2}} = 0, \quad (\text{A.21a})$$

$$\left[\tilde{D} - \sqrt{-\omega}\right] \left[\tilde{D} + \sqrt{-\omega}\right] \frac{x}{\sqrt{1 + \lambda x^2}} = 0, \quad (\text{A.21b})$$

where $\tilde{D} = (1 + \lambda x^2)D$. The above equation can be written as

$$\frac{\tilde{D} \left[\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - \frac{\sqrt{-\omega} x}{\sqrt{1 + \lambda x^2}} \right]}{\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - \frac{\sqrt{-\omega} x}{\sqrt{1 + \lambda x^2}}} + \frac{\tilde{D} \left[\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] + \frac{\sqrt{-\omega} x}{\sqrt{1 + \lambda x^2}} \right]}{\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] + \frac{\sqrt{-\omega} x}{\sqrt{1 + \lambda x^2}}} = 0. \quad (\text{A.22})$$

Integrating Eq. (A.22), we get

$$I_1 = \frac{\dot{x}^2 + \omega x^2}{1 + \lambda x^2}, \quad (\text{A.23})$$

where I_1 is the first integral for the scalar ML oscillator equation [14].

Appendix B. Factorization of scalar case corresponding to Eq. (2)

Now, we consider the scalar mixed [24, 25] case by including an \dot{x} term to Eq. (A.1), that is

$$\ddot{x} + h(x)\dot{x}^2 + f(x)\dot{x} + g(x) = 0, \quad (\text{B.1})$$

where $f(x)$, $g(x)$ and $h(x)$ are arbitrary functions of x , and discuss how to get the factorized form for Eq. (B.1). For this purpose, we follow the same procedure used in Appendix A earlier. To start with, we assume that Eq. (B.1) can be factorized in the form given by Eq. (A.3). Now, expanding it and comparing the resultant equation with the coefficients of various powers of \dot{x} of Eq. (B.1), we get

$$(\phi_2\psi_x)_x - h\phi_2\psi_x = 0, \quad (\text{B.2})$$

$$\phi_1\tilde{\phi}_3\psi - \phi_2\psi_x g = 0, \quad (\text{B.3})$$

and

$$(\phi_1\psi)_x + \phi_2\tilde{\phi}_3\psi_x + f\phi_2\psi_x = 0. \quad (\text{B.4})$$

It is to be noted that Eqs. (B.2) and (B.3) are exactly the same as Eqs. (A.4) and (A.5), respectively, while Eq. (B.4) differs from Eq. (A.6). Hence, we can deduce the forms of ϕ_2 and $\tilde{\phi}_3$ to be the same as given by Eq. (A.7) and (A.8), provided ϕ_1 and ψ are known. The form of the product function $(\phi_1\psi)$ can be fixed by solving Eq. (B.4). Now, using Eqs. (A.7) and (A.8) in Eq. (B.4), we get

$$(\phi_1\psi)_x + \frac{g c_1^2 e^{2\int h dx}}{\phi_1\psi} + f c_1 e^{\int h dx} = 0. \quad (\text{B.5})$$

Eq. (B.5) is of the form of the Abel equation of the second kind, that is

$$\xi\xi' + F(x)\xi + G(x) = 0, \quad ' \equiv \frac{d}{dx}. \quad (\text{B.6})$$

A general condition for the separability [26] of Eq. (B.6) is known as

$$\left(\frac{G}{F}\right)' = \delta F, \quad (\text{B.7})$$

where δ is an arbitrary constant. Under this condition and with the change of dependent variable $\xi = \left(\frac{G}{F}\right) \frac{1}{w}$, we get the separable equation in the form

$$w' = \frac{F^2}{G} w(w^2 + w + \delta), \quad (\text{B.8})$$

which is integrable. Hence, we conclude that equations belonging to Eq. (B.1) can be factorized into the form (A.2) if Eq. (B.5) can be solved.

As an example to this class of equations, we consider the damped ML oscillator [27], that is

$$\ddot{x} - \frac{\lambda x}{1 + \lambda x^2} \dot{x}^2 + \frac{\alpha}{1 + \lambda x^2} \dot{x} + \frac{\lambda_1 x}{1 + \lambda x^2} = 0, \quad (\text{B.9})$$

where λ , α and λ_1 are arbitrary parameters. Comparing Eqs. (B.9) and (B.1), we get

$$h = -\frac{\lambda x}{1 + \lambda x^2}, \quad f = \frac{\alpha}{1 + \lambda x^2}, \quad g = \frac{\lambda_1 x}{1 + \lambda x^2}, \quad (\text{B.10})$$

which indeed satisfy the separability condition (B.7). Now, rewriting Eq. (B.5) for this case, we get

$$(\phi_1 \psi)(\phi_1 \psi)_x + \frac{c_1 \alpha}{(1 + \lambda x^2)^{\frac{3}{2}}} (\phi_1 \psi) + \frac{c_1^2 \lambda_1 x}{(1 + \lambda x^2)^2} = 0. \quad (\text{B.11})$$

A particular solution to the above equation can be written as

$$\phi_1 \psi = \frac{a \sqrt{\lambda_1} x}{\sqrt{1 + \lambda x^2}}, \quad (\text{B.12})$$

where we have considered $c_1 = 1$ and a is defined by the relation $\alpha = -\frac{\sqrt{\lambda_1}(a^2+1)}{a}$. Now, we consider ψ as $\frac{x}{1+\lambda x^2}$, then ϕ_1 can be fixed as $\phi_1 = a \sqrt{\lambda_1}$. The remaining functions, that is ϕ_2 and $\tilde{\phi}_3$ can be obtained from Eqs. (A.7) and (A.8) as

$$\phi_2 = 1 + \lambda x^2, \quad (\text{B.13})$$

and

$$\phi_3 = \frac{\sqrt{\lambda_1}}{a(1 + \lambda x^2)}. \quad (\text{B.14})$$

Substituting the obtained forms of the functions ϕ_1 , ϕ_2 , $\tilde{\phi}_3$ and ψ in Eq. (A.3) and rewriting the resultant expression, we get the factorized form for Eq. (B.1) as

$$\left[(1 + \lambda x^2)D - \frac{\sqrt{\lambda_1}}{a} \right] \left[(1 + \lambda x^2)D - a \sqrt{\lambda_1} \right] \frac{x}{\sqrt{1 + \lambda x^2}} = 0. \quad (\text{B.15})$$

Similarly, choosing another particular solution $\phi_1 \psi = \frac{\sqrt{\lambda_1} x}{a \sqrt{1 + \lambda x^2}}$ to Eq. (B.11), we can write Eq. (B.9) as

$$\left[(1 + \lambda x^2)D - \sqrt{\lambda_1} a \right] \left[(1 + \lambda x^2)D - \frac{\sqrt{\lambda_1}}{a} \right] \frac{x}{\sqrt{1 + \lambda x^2}} = 0. \quad (\text{B.16})$$

To prove the integrability of the damped ML oscillator (B.9) we follow the same procedure as that of the scalar ML oscillator case and we obtain

$$\frac{a}{\sqrt{\lambda_1}} \frac{\tilde{D} \left[\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - a \sqrt{\lambda_1} \frac{x}{\sqrt{1 + \lambda x^2}} \right]}{\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - a \sqrt{\lambda_1} \frac{x}{\sqrt{1 + \lambda x^2}}} - \frac{1}{a \sqrt{\lambda_1}} \frac{\tilde{D} \left[\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - \frac{\sqrt{\lambda_1}}{a} \frac{x}{\sqrt{1 + \lambda x^2}} \right]}{\tilde{D} \left[\frac{x}{\sqrt{1 + \lambda x^2}} \right] - \frac{\sqrt{\lambda_1}}{a} \frac{x}{\sqrt{1 + \lambda x^2}}} = 0, \quad (\text{B.17})$$

where $\tilde{D} = (1 + \lambda x^2)D$. Integrating Eq. (B.17), we get

$$I_1 = \left[\frac{\dot{x} - a \sqrt{\lambda_1} x}{\sqrt{1 + \lambda x^2}} \right]^a \left[\frac{a \dot{x} - \sqrt{\lambda_1} x}{a \sqrt{1 + \lambda x^2}} \right]^{-\frac{1}{a}}, \quad (\text{B.18})$$

where I_1 is the first integral for the scalar damped ML oscillator equation as shown in Ref. [27].

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